Supersymmetric Hamilton Operator and Entanglement

Willi-Hans Steeb and Yorick Hardy

International School for Scientific Computing, University of Johannesburg, Auckland Park 2006, South Africa

Reprint requests to Prof. W.-H. S; E-mail: WHS@NA.RAU.AC.ZA

Z. Naturforsch. **61a,** 139 – 140 (2006); received February 22, 2006

We study the entanglement of Fermi particles of a supersymmetric Hamilton operator given by a simple Fermi-Bose system.

Key words: Supersymmetric Hamiltonian; Entanglement; Fermi Operators; Bose Operators.

Entanglement has been studied in detail for finite-dimensional quantum systems and to a lesser extent for infinite-dimensional quantum systems (see [1, 2] and references therein). Here we study the entanglement for states of a supersymmetric Hamilton operator [3] given by Bose operators $b^\dagger$, b and Fermi operators with spin up and spin down, i.e. $c_\uparrow^\dagger$, $c_\downarrow^\dagger$, $c_\uparrow$, $c_\downarrow$. Let

$$Q := b \otimes c_\uparrow^\dagger c_\downarrow^\dagger \tag{1}$$

be a linear operator, where b is a Bose annihilation operator, $c_\uparrow^\dagger$ is a Fermi creation operator with spin up, $c_\downarrow^\dagger$ is a Fermi operator with spin down and $\otimes$ the tensor product [4]. Since $c_\sigma^\dagger c_\sigma^\dagger = 0$, $\sigma \in \{\uparrow, \downarrow\}$ we find that $Q^2 = 0$. We define the supersymmetric Hamilton operator $\hat{H}$ as

$$\hat{H} := [Q, Q^\dagger]_+ \equiv QQ^\dagger + Q^\dagger Q.$$

From (1) we obtain $Q^\dagger = b^\dagger \otimes c_\downarrow c_\uparrow$. Let $\hat{n}_B := b^\dagger b$, $\hat{n}_\uparrow := c_\uparrow^\dagger c_\uparrow$, $\hat{n}_\downarrow := c_\downarrow^\dagger c_\downarrow$ be the number operators. Applying $[b, b^\dagger] = I_\text{B}$ and $[c_\sigma, c_{\sigma'}^\dagger]_+ = I_\text{F}\delta_{\sigma,\sigma'}$ we arrive at

$$\hat{H} = (2\hat{n}_B + I_B) \otimes \hat{n}_\uparrow \hat{n}_\downarrow + \hat{n}_B \otimes (I_F - \hat{n}_\uparrow - \hat{n}_\downarrow),$$

where I_B is the identity operator in the Hilbert space $\mathcal{H}_B$ of the Bose operators and I_F is the identity operator in the Hilbert space $\mathcal{H}_F$ of the Fermi operators. Straightforward calculation yields $[\hat{H}, Q] = 0$ and $[\hat{H}, Q^\dagger Q] = 0$. Thus the three operators $\hat{H}$, Q, $Q^\dagger Q$ may be diagonalized simultaneously. Let $|n\rangle$ be the number states (Fock states), where $n = 0, 1, 2, \ldots$ and $\langle n|n\rangle = 1$. For the Fermi operators we use the matrix representation [4]

$$c_\uparrow^\dagger = \frac{1}{2}\sigma_+ \otimes I_2, \qquad c_\downarrow^\dagger = \frac{1}{2}\sigma_z \otimes \sigma_+.$$

Thus

$$c_\uparrow = \frac{1}{2}\sigma_- \otimes I_2, \qquad c_\downarrow = \frac{1}{2}\sigma_z \otimes \sigma_-$$

and

$$c_\uparrow^\dagger c_\downarrow^\dagger = -\frac{1}{4}\sigma_+ \otimes \sigma_+.$$

Thus the Fermi operators act in the Hilbert space $\mathbf{C}^4$. It follows that

$$\hat{n}_\uparrow = c_\uparrow^\dagger c_\uparrow = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2, \qquad \hat{n}_\downarrow = c_\downarrow^\dagger c_\downarrow = I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\hat{n}_\uparrow \hat{n}_\downarrow = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then a basis in the product Hilbert space is given by

$$|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0}, \quad |n\rangle \otimes \binom{1}{0} \otimes \binom{0}{1},$$

$$|n\rangle \otimes \binom{0}{1} \otimes \binom{1}{0}, \quad |n\rangle \otimes \binom{0}{1} \otimes \binom{0}{1},$$

where $n = 0, 1, 2, \ldots$. Now we obtain

$$\hat{H}|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0} = (n+1)|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0}.$$

This is an eigenvalue equation with eigenvalue $n+1$, where $n = 0, 1, 2, \ldots$. Furthermore

$$\hat{H}|n\rangle \otimes \binom{1}{0} \otimes \binom{0}{1} = 0,$$

$$\hat{H}|n\rangle \otimes \binom{0}{1} \otimes \binom{1}{0} = 0.$$

Both states have eigenvalue 0. Finally

$$\hat{H}|n\rangle \otimes \binom{0}{1} \otimes \binom{0}{1} = n|n\rangle \otimes \binom{0}{1} \otimes \binom{0}{1}$$

with eigenvalue n, where $n = 0, 1, 2, \ldots$. Thus the lowest eigenvalue is 0. The Bell states are given by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(\binom{1}{0} \otimes \binom{1}{0} + \binom{0}{1} \otimes \binom{0}{1} \right),$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} \left(\binom{1}{0} \otimes \binom{1}{0} - \binom{0}{1} \otimes \binom{0}{1} \right),$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} \left(\binom{1}{0} \otimes \binom{0}{1} + \binom{0}{1} \otimes \binom{1}{0} \right),$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} \left(\binom{1}{0} \otimes \binom{0}{1} - \binom{0}{1} \otimes \binom{1}{0} \right).$$

Consider the product states of the number states and the Bell states. Applying the Hamilton operator we find

$$\hat{H}|n\rangle \otimes |\Phi^+\rangle = n|n\rangle \otimes |\Phi^+\rangle + \frac{1}{\sqrt{2}}|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0},$$

$$\hat{H}|n\rangle \otimes |\Phi^-\rangle = n|n\rangle \otimes |\Phi^-\rangle + \frac{1}{\sqrt{2}}|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0},$$

$$\hat{H}|n\rangle \otimes |\Psi^+\rangle = 0,$$

$$\hat{H}|n\rangle \otimes |\Psi^-\rangle = 0.$$

Consider now the unitary operator $U(t) = \exp(-\mathrm{i}\hat{H}t)$. Then we obtain

$$U(t)|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0} = \mathrm{e}^{-\mathrm{i}t(n+1)}|n\rangle \otimes \binom{1}{0} \otimes \binom{1}{0},$$

$$U(t)|n\rangle \otimes \binom{0}{1} \otimes \binom{0}{1} = \mathrm{e}^{-\mathrm{i}tn}|n\rangle \otimes \binom{0}{1} \otimes \binom{0}{1}$$

and

$$U(t)|n\rangle \otimes \binom{1}{0} \otimes \binom{0}{1} = |n\rangle \otimes \binom{1}{0} \otimes \binom{0}{1},$$

$$U(t)|n\rangle \otimes \binom{0}{1} \otimes \binom{1}{0} = |n\rangle \otimes \binom{0}{1} \otimes \binom{1}{0}.$$

It follows that

$$U(t)|n\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}tn}|n\rangle \otimes \left(\mathrm{e}^{-\mathrm{i}t} \binom{1}{0} \otimes \binom{1}{0} + \binom{0}{1} \otimes \binom{0}{1} \right),$$

$$U(t)|n\rangle \otimes |\Phi^-\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}tn}|n\rangle \otimes \left(\mathrm{e}^{-\mathrm{i}t} \binom{1}{0} \otimes \binom{1}{0} - \binom{0}{1} \otimes \binom{0}{1} \right),$$

and $U(t)|n\rangle \otimes |\Psi^+\rangle = |n\rangle \otimes |\Psi^+\rangle$, $U(t)|n\rangle \otimes |\Psi^-\rangle = |n\rangle \otimes |\Psi^-\rangle$. Thus the states $|n\rangle \otimes |\Psi^+\rangle$ and $|n\rangle \otimes |\Psi^-\rangle$ do not change under the unitary transformation. The Fermi part of the state $U(t)|n\rangle \otimes |\Phi^\pm\rangle$ is also a Bell state. There is a continuous oscillation between $|n\rangle \otimes |\Phi^+\rangle$ and $|n\rangle \otimes |\Phi^-\rangle$ with periodicity 2π.

[1] W.-H. Steeb and Y. Hardy, Problems and Solutions in Quantum Computing and Quantum Information, World Scientific, Singapore 2006.
[2] Y. Hardy and W.-H. Steeb, Int. J. Theor. Phys. **43**, 2207 (2004).
[3] W.-H. Steeb, Problems and Solutions in Theoretical and Mathematical Physics, Vol. II: Advanced Level, 2nd ed., World Scientific, Singapore 2003.
[4] W.-H. Steeb, Matrix Calculus and Kronecker Product with Applications and C++ Programs, World Scientific, Singapore 1997.